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TECHNICAL STABILITY OF NONLINEAR STATES OF AN ELASTIC VEHICLE IN VERTICAL FLIGHT

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Sufficient conditions of technical stability of nonlinear dynamic states of extended elastic flying systems in controlled longitudinal vertical flight are obtained. In these flying systems, the effect of variation of their cross-sectional area, transverse strains, and oscillations is taken into account. The formulated criteria of technical stability depend on the basic parameters of the process controlled, namely, on the increment of the transverse load due to the curvature of the axis of the system and aerodynamic forces during vertical flight.

Introduction. This work deals with technical stability [1–8] of nonlinear dynamic states of extended elastic rocket-type flying systems moving in the vertical plane. These systems are shaped as thin extended bodies with a variable cross section where high transverse strains and oscillations appear during the flight. With increasing size, such dynamic bodies become less rigid, and the influence of elastic and other oscillations on flight and flight control becomes significant. Interaction of strain, angular motion of the body of the system, external aerodynamic forces, and internal hydrodynamic disturbances caused by the oscillating liquid in the tanks of the system can lead to undesirable effects, such as self-induced oscillations, loss of stability, etc. As a result, the flying system may fail to fly along a given trajectory. Flight control serves to reduce deviations from given angular and other motions of the flying vehicle. Flight control can stabilize or, vice versa, if an improper control action is chosen, shake the liquid in the tanks and increase elastic oscillations. Sufficient conditions of technical stability of a given dynamic system within finite and infinite time intervals are obtained for a proposed flight-control mode. The method of comparison based on optimization of distributed processes is used in combination with the Lyapunov direct method. The study performed is based on the results of [9–18].

Formulation of a Controlled Boundary-Value Problem of the System. Let the flying system be an extended elastic body, for instance, a thin body of revolution or a body of revolution with low-aspect-ratio wings and finning, which moves in the vertical plane [9–11]. Flight conditions require small deviations of the longitudinal axis of the system from a given motion. Each point of the longitudinal axis of the flying vehicle should move along a certain trajectory. During the flight, however, there are always deviations from a given motion. Appropriate flight control should ensure a small deviation of the trajectory from a given one, for example, a straight line, and the greatest possible flight accuracy. The flight velocity is assumed to be constant. Oscillations of the axis of the flying system as a variable-section beam under the action of elastic forces, weight, and aerodynamic forces are described by the following equations [7, 9–11, 14–16]:

$$\frac{\partial\varphi_1}{\partial t} = \varphi_2, \qquad \frac{\partial\varphi_2}{\partial t} = L(\varphi) + \frac{\alpha}{m}\,u, \qquad t \in T_1, \quad x \in D. \tag{1}$$

Here

$$L(\varphi) = -\frac{1}{m}\,\frac{\partial^2}{\partial x^2}\Big(EI\,\frac{\partial^2\varphi_1}{\partial x^2}\Big) - \frac{a_1}{m}\,\varphi_2 - \frac{b_1}{m}\,\frac{\partial\varphi_1}{\partial x} - \frac{1}{m}\,\bar{Q},$$

$x = \tilde{x}/l$, $\varphi_1 = \tilde{\varphi}/l$, $\varphi_2 = (1/\sqrt{gl})(\partial\tilde{\varphi}/\partial\tau)$, $t = \sqrt{g/l}\,\tau$, $EI = \tilde{E}\tilde{I}/(Gl^2)$, $a_1 = \tilde{a}l\sqrt{gl}/G$, $b_1 = \tilde{b}l/G$, $m = \tilde{m}gl/G$, $0 < \mu \leqslant \mu_0 < 1$, $T_1 = [t_0, N\mu^{-1}]$, $D \equiv (0,1)$, $t_0 = \text{const} \geqslant 0$, $N = \text{const} > 0$, $T_1 \subset I_1 \equiv [t_0, +\infty)$, $u = u(x,t)$ is

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flight control, $\alpha = \alpha(x)$ is a given function that takes into account the point of application of the governing forces, for example, if the flight control is applied at the sector $[a, 1]$ of the axis only, then we assume that $\alpha(x) \equiv 0$ for $x \in [0, a)$ and $\alpha(x) = 1$ for $x \in [a, 1]$), G is the weight, l is the length of the flying body, $\tilde{x}$ is the coordinate of the current cross section, $\tilde{\varphi} = \tilde{\varphi}(\tilde{x}, \tau)$ is the deviation of the axis from the equilibrium state, φ_1 is the dimensionless deviation of the axis, $\tilde{E}\tilde{I}$ is the flexural rigidity, $\tilde{m}$ is the mass per unit length, $\tilde{a}$ and $\tilde{b}$ are the coefficients of aerodynamic forces, g is the acceleration of gravity, $\bar{Q}$ is the increment of the transverse load due to the curvature of the longitudinal axis of the system, τ is the time, and t is the dimensionless time. In the case of horizontal flight, we have $\bar{Q} \equiv 0$ [9, 10, 14]. Note, the change in transverse forces due to the force of gravity at small deviations from the equilibrium state is a quantity of the second order of smallness [14]. If the flight is close to vertical, the moment of the force of gravity is [11, 14]

$$M_g = \int_0^{\tilde{x}} \tilde{m} g[\tilde{\varphi}(\tilde{x}, \tau) - \tilde{\varphi}(\xi, \tau)]\, d\xi.$$

Therefore, for a distributed load, we find

$$Q_0 = \frac{\partial^2 M_g}{\partial \tilde{x}^2} = \frac{\partial}{\partial \tilde{x}}\Big(q_0 \,\frac{\partial \tilde{\varphi}}{\partial \tilde{x}}\Big), \qquad q_0 = q_0(\tilde{x}) = \int_0^{\tilde{x}} g\tilde{m}(\xi)\, d\xi.$$

In dimensionless variables, for vertical flight, we obtain [13]

$$\bar{Q} = \frac{m}{\tilde{m} g}\, Q_0 = \frac{\partial}{\partial x}\Big(q\, \frac{\partial \varphi_1}{\partial x}\Big), \qquad q = \frac{q_0}{G} = \int_0^x m\, dx. \tag{2}$$

The coefficients a_1 and b_1 are found by solving aerodynamics equations [12]. In a flight with a supersonic velocity, the law of planar cross sections is valid [7, 15], and the aerodynamic forces are more readily found [12, 13]. The flow pressure in a transverse flow past thin bodies is determined by the local angle of attack

$$\tilde{\alpha} = \frac{1}{v_\infty} \frac{\partial \tilde{\varphi}}{\partial \tau} + \frac{\partial \tilde{\varphi}}{\partial \tilde{x}},$$

where v_∞ is the free-stream velocity.

Let $c = c(\tilde{x})$ be the span of the thin wing in a cross section with the coordinate $\tilde{x}$. The normal component of the aerodynamic force in this cross section has the form

$$n(\tilde{x}, t) = -2 \int_{-c(\tilde{x})/2}^{c(\tilde{x})/2} (p - p_\infty)\, dx = \frac{2\chi p_\infty v_\infty c(\tilde{x})}{a_\infty}\, \tilde{\alpha},$$

where $\chi = c_p/c_V$ (c_p and c_V are the heat capacities at constant pressure p and volume V, respectively) and p_∞ and a_∞ are the free-stream pressure and velocity of sound. Then, the coefficients $\tilde{a}$ and $\tilde{b}$ become [9–11]

$$\tilde{a} = 2\chi p_\infty c(\tilde{x})/a_\infty, \qquad \tilde{b} = 2\chi p_\infty v_\infty c(\tilde{x})/a_\infty.$$

For wings of rectangular planform, $\tilde{a}$ and $\tilde{b}$ are constant. The normal component of aerodynamic forces in the flow past extended bodies of revolution is [9–14, 16]

$$n(\tilde{x}, t) = \rho_\infty v_\infty^2 R \,\frac{dR}{d\tilde{x}} \tilde{\alpha},$$

where $R = R(\tilde{x})$ is the radius of the body of revolution and ρ_∞ is the free-stream density. In this case, we have

$$\tilde{a} = \rho_\infty v_\infty R \,\frac{dR}{d\tilde{x}}, \qquad \tilde{b} = \rho_\infty v_\infty^2 R \,\frac{dR}{d\tilde{x}}.$$

For $R = R_0\sqrt{\tilde{x}}$, the coefficients $\tilde{a}$ and $\tilde{b}$ are independent of $\tilde{x}$.

In what follows, we consider the case where $\bar{Q}$ is determined by Eq. (2), i.e., the flight is close to vertical. We introduce boundary conditions for the function $\varphi_1 = \varphi_1(x, t)$. The moment and concentrated force are absent at the leading edge of the body ($x = 0$):

$$\Big(\frac{\partial^2 \varphi_1}{\partial x^2}\Big)_{x=0} = \Big[\frac{\partial}{\partial x}\Big(EI\, \frac{\partial^2 \varphi_1}{\partial x^2}\Big)\Big]_{x=0} = 0. \tag{3}$$

The moment is absent near the trailing edge ($x = 1$):

$$\left(\frac{\partial^2 \varphi_1}{\partial x^2}\right)_{x=1} = 0. \tag{4}$$

The motion of the flying system can be controlled, in addition to the control u, by the control force u_S applied at the point $x = 1$, i.e.,

$$\left[\frac{\partial}{\partial x}\left(EI\,\frac{\partial^2 \varphi_1}{\partial x^2}\right)\right]_{x=1} = u_S. \tag{5}$$

Let the initial conditions of a prescribed process be defined:

$$\varphi_1(x,t)|_{t=t_0} = \omega_0(x), \qquad \varphi_2(x,t)|_{t=t_0} = v_0(x), \qquad t_0 \in T_1, \qquad x \in D,$$
$$\varphi_0(x) \equiv (\omega_0(x), v_0(x))^{\mathrm{t}}. \tag{6}$$

We study the boundary-value problem of control (1)–(6) under the assumption that this system has a unique solution in the class of functions continuous in terms of t and$x$ and possessing continuous derivatives with respect to t and x of necessary orders in the case of prescribed functions $\omega_0(x)$ and $v_0(x)$ that satisfy the necessary conditions of matching at the system boundary [3–7, 15]. The measure $\rho = \rho(\varphi)$ that characterizes the deviation of the functions $\varphi = (\varphi_1(x,t), \varphi_2(x,t))$ $[\varphi_2(x,t) = \partial\varphi_1(x,t)/\partial t]$ from the value $\varphi = 0$ of the undisturbed process is chosen to be [2–8]

$$\rho[\varphi] = \int_0^1 \left[\left(\frac{\partial^2 \varphi_1}{\partial x^2}\right)^2 + \left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \varphi_1^2 + \varphi_2^2\right] dx. \tag{7}$$

Let the domain of possible initial states Ω_0 of system (1)–(6) be defined in the form

$$\Omega_0 = \{\varphi\colon \quad \rho \leqslant \tilde{a}_1,\ \tilde{a}_1 > 0\}$$

and the domain of admissible current states $\Omega(t)$ of system (1)–(6) be defined in the form

$$\Omega(t) = \{\varphi\colon \quad \rho \leqslant \eta(t),\ 0 < \eta(t) \leqslant \tilde{\eta},\ \tilde{\eta} = \text{const} > 0\},$$

where $\tilde{a}_1$ is a given number and $\eta(t)$ is a function bounded in the range $T_1 \subset I_1$. In this case, the conditions

$$\tilde{a}_1 \leqslant \eta(t_0), \qquad \Omega_0 \subset \Omega(t_0), \qquad t_0 \in T_1$$

are valid.

The optimal control minimizing the functional

$$J_0 = \int_0^T W\,d\tau + W_0 \qquad (T = N\mu^{-1}),$$

where

$$W = \int_0^1\int_0^1 \sum_{i=1}^2 w_{ii}(x,\xi)\varphi_i(x,t)\varphi_i(\xi,t)\,dx\,d\xi + \int_0^1 \omega(x)u^2\,dx + \omega_S u_S^2,$$

$$W_0 = \int_0^1\int_0^1 \sum_{i=1}^2 \omega_{ii}(x,\xi)\varphi_i(x,T)\varphi_i(\xi,T)\,dx\,d\xi,$$

is determined by the method of dynamic programming [9–11]. Here $w_{11} = w_{11}(x,\xi)$, $w_{22} = w_{22}(x,\xi)$, $\omega = \omega(x)$, $\omega_{11} = \omega_{11}(x,\xi)$, and $\omega_{22} = \omega_{22}(x,\xi)$ are prescribed weight functions, and $\omega(1) = \omega_S$.

To find the optimal functional V_0 of the basic equation of dynamic programming

$$V_0 = \int_0^1\int_0^1 \sum_{i,j=1}^2 v_{ij}(x,\xi,t)\varphi_i(x,t)\varphi_j(\xi,t)\,dx\,d\xi$$

we have the following relation [11, 14]:

$$
\begin{aligned}
K = \frac{dV_0}{dt} + W = \int_0^1\int_0^1 \Big\{ \Big[\frac{\partial v_{11}}{\partial t} - \frac{\partial^2}{\partial x^2}\Big(EI\,\frac{\partial^2 (v_{21}/m)}{\partial x^2}\Big) - \frac{\partial}{\partial x}\Big(q\,\frac{\partial (v_{21}/m)}{\partial x}\Big) + \frac{\partial (b_1 v_{21}/m)}{\partial x} \\
- \frac{\partial^2}{\partial \xi^2}\Big(EI\,\frac{\partial^2 (v_{12}/m)}{\partial \xi^2}\Big) - \frac{\partial}{\partial \xi}\Big(q\,\frac{\partial (v_{12}/m)}{\partial \xi}\Big) + \frac{\partial (b_1 v_{12}/m)}{\partial \xi} + \omega_{11}\Big]\varphi_1(x,t)\varphi_1(\xi,t) \\
+ \Big[\frac{\partial v_{12}}{\partial t} + v_{11}(x,\xi,t) - \frac{\partial^2}{\partial x^2}\Big(EI\,\frac{\partial^2 (v_{22}/m)}{\partial x^2}\Big) - \frac{\partial}{\partial x}\Big(q\,\frac{\partial (v_{22}/m)}{\partial x}\Big) \\
+ \frac{\partial (b_1 v_{22}/m)}{\partial x} - \frac{a_1(\xi)}{m(\xi)}\,v_{12}(x,\xi,t)\Big]\varphi_1(x,t)\varphi_2(\xi,t) \\
+ \Big[\frac{\partial v_{21}}{\partial t} + v_{11}(x,\xi,t) - \frac{\partial^2}{\partial \xi^2}\Big(EI\,\frac{\partial^2 (v_{22}/m)}{\partial \xi^2}\Big) - \frac{\partial}{\partial \xi}\Big(q\,\frac{\partial (v_{22}/m)}{\partial \xi}\Big) \\
+ \frac{\partial (b_1 v_{22}/m)}{\partial \xi} - \frac{a_1(x)}{m(x)}\,v_{21}(x,\xi,t)\Big]\varphi_2(x,t)\varphi_1(\xi,t) \\
+ \Big[\frac{\partial v_{22}}{\partial t} + v_{12}(x,\xi,t) + v_{21}(x,\xi,t) - v_{22}(x,\xi,t)\Big(\frac{a_1(x)}{m(x)} + \frac{a_1(\xi)}{m(\xi)}\Big) + w_{22}\Big]\varphi_2(x,t)\varphi_2(\xi,t)\Big\}\,dx\,d\xi \\
- \int_0^1 \Big\{\Big[EI(\xi)\,\frac{\partial^2 (v_{12}/m)}{\partial \xi^2} + \frac{v_{12}(x,\xi,t)q(\xi)}{m(\xi)}\Big]\frac{\partial \varphi_1}{\partial \xi} \\
+ \Big[-\frac{\partial}{\partial \xi}\Big(EI\,\frac{\partial^2 (v_{12}/m)}{\partial \xi^2}\Big) + \frac{b_1(\xi)}{m(\xi)}\,v_{12}(x,\xi,t) - q(\xi)\,\frac{\partial (v_{12}/m)}{\partial \xi}\Big]\varphi_1(\xi,t)\Big\}_{\xi=0}^{\xi=1}\varphi_1(x,t)\,dx \\
- \int_0^1 \Big\{\Big[EI(\xi)\,\frac{\partial^2 (v_{22}/m)}{\partial \xi^2} + \frac{v_{22}(x,\xi,t)q(\xi)}{m(\xi)}\Big]\frac{\partial \varphi_1}{\partial \xi} \\
+ \Big[-\frac{\partial}{\partial \xi}\Big(EI\,\frac{\partial^2 (v_{22}/m)}{\partial \xi^2}\Big) + \frac{b_1(\xi)}{m(\xi)}\,v_{22}(x,\xi,t) - q(\xi)\,\frac{\partial (v_{22}/m)}{\partial \xi}\Big]\varphi_1(\xi,t)\Big\}_{\xi=0}^{\xi=1}\varphi_2(x,t)\,dx \\
- \int_0^1 \Big\{\Big[EI(x)\,\frac{\partial^2 (v_{21}/m)}{\partial x^2} + \frac{v_{21}(x,\xi,t)q(x)}{m(x)}\Big]\frac{\partial \varphi_1}{\partial x} \\
+ \Big[-\frac{\partial}{\partial x}\Big(EI\,\frac{\partial^2 (v_{21}/m)}{\partial x^2}\Big) + \frac{b_1(x)}{m(x)}\,v_{21}(x,\xi,t) - q(x)\,\frac{\partial (v_{12}/m)}{\partial x}\Big]\varphi_1(x,t)\Big\}_{x=0}^{x=1}\varphi_1(\xi,t)\,d\xi \\
- \int_0^1 \Big\{\Big[EI(x)\,\frac{\partial^2 (v_{22}/m)}{\partial x^2} + \frac{v_{22}(x,\xi,t)q(x)}{m(x)}\Big]\frac{\partial \varphi_1}{\partial x} \\
+ \Big[-\frac{\partial}{\partial x}\Big(EI\,\frac{\partial^2 (v_{22}/m)}{\partial x^2}\Big) + \frac{b_1(x)}{m(x)}\,v_{22}(x,\xi,t) - q(x)\,\frac{\partial (v_{22}/m)}{\partial x}\Big]\varphi_1(x,t)\Big\}_{x=0}^{x=1}\varphi_2(\xi,t)\,d\xi \\
+ \int_0^1 \omega(x)u^2(x,t)\,dx + \int_0^1 \frac{\alpha(x)u(x,t)}{m(x)}\Big\{\int_0^1 [v_{12}(\xi,x,t) + v_{21}(x,\xi,t)]\varphi_1(\xi,t)\,d\xi \\
+ \int_0^1 [v_{22}(x,\xi,t) + v_{22}(\xi,x,t)]\varphi_2(\xi,t)\,d\xi\Big\}\,dx.
\end{aligned}
$$

The functions $v_{ij} = v_{ij}(x, \xi, t)$ should satisfy the final conditions

$$v_{ij}(x, \xi, T) = 0, \quad i \neq j, \qquad v_{ii}(x, \xi, T) = \omega_{ii}(x, \xi) \quad (i = 1,\, 2) \tag{8}$$

and the boundary conditions

$$\begin{gathered}
\Big[EI(\xi)\,\frac{\partial^2 (v_{12}/m)}{\partial \xi^2} + \frac{q(\xi) v_{12}(x, \xi, t)}{m(\xi)}\Big]_{\xi=0,\ \xi=1} = 0, \\
\Big[-\frac{\partial}{\partial \xi}\Big(EI\,\frac{\partial^2 (v_{12}/m)}{\partial \xi^2}\Big) - q(\xi)\,\frac{\partial (v_{12}/m)}{\partial \xi} + \frac{b_1(\xi) v_{12}(x, \xi, t)}{m(\xi)}\Big]_{\xi=0,\ \xi=1} = 0, \\
\Big[EI(x)\frac{\partial^2 (v_{21}/m)}{\partial x^2} + \frac{q(x) v_{12}(x, \xi, t)}{m(x)}\Big]_{x=0,\ x=1} = 0, \\
\Big[-\frac{\partial}{\partial x}\Big(EI\,\frac{\partial^2 (v_{21}/m)}{\partial x^2}\Big) - q(x)\,\frac{\partial (v_{21}/m)}{\partial x} + \frac{b_1(x) v_{21}(x, \xi, t)}{m(x)}\Big]_{x=0,\ x=1} = 0, \\
\Big[EI(\xi)\,\frac{\partial^2 (v_{22}/m)}{\partial \xi^2} + \frac{q(\xi) v_{22}(x, \xi, t)}{m(\xi)}\Big]_{\xi=0,\ \xi=1} = 0, \\
\Big[-\frac{\partial}{\partial \xi}\Big(EI\,\frac{\partial^2 (v_{22}/m)}{\partial \xi^2}\Big) - q(\xi)\,\frac{\partial (v_{22}/m)}{\partial \xi} + \frac{b_1(\xi) v_{22}(x, \xi, t)}{m(\xi)}\Big]_{\xi=0,\ \xi=1} = 0, \\
\Big[EI(x)\,\frac{\partial^2 (v_{22}/m)}{\partial x^2} + \frac{q(x) v_{22}(x, \xi, t)}{m(x)}\Big]_{x=0,\ x=1} = 0, \\
\Big[-\frac{\partial}{\partial x}\Big(EI\,\frac{\partial^2 (v_{22}/m)}{\partial x^2}\Big) - q(x)\,\frac{\partial (v_{22}/m)}{\partial x} + \frac{b_1(x) v_{22}(x, \xi, t)}{m(x)}\Big]_{x=0,\ x=1} = 0.
\end{gathered} \tag{9}$$

The condition $\min\limits_u(K)$ yields the optimal control

$$u = -\frac{\alpha(x)}{2\omega(x) m(x)} \int\limits_0^1 \sum_{i=1}^{2} [v_{i2}(\xi, x, t) + v_{2i}(x, \xi, t)]\varphi_i(\xi, t)\, d\xi; \tag{10}$$

$$u_S = -\frac{1}{2\omega_S m(1)} \int\limits_0^1 \sum_{i=1}^{2} [v_{i2}(\xi, 1, t) + v_{2i}(1, \xi, t)]\varphi_i(\xi, t)\, d\xi. \tag{11}$$

Here the factors at the functions $\varphi_1(\xi, t)$ and $\varphi_2(\xi, t)$ are feedback amplification coefficients, which are functions of coordinates of the points of the body axis. Displacements of different points make different contributions to the magnitude of control, depending on the place where the point of the axis is located [11].

Expressions (10) and (11) are regulator equations closing system (1)–(6) and are a linear operator on the set of shapes of axis deviations from the straight line and their velocities. To solve (10), (11), we have to measure the values of $\varphi_1(\xi, t)$ and $\varphi_2(\xi, t)$ at each point of the axis at each time. For the measured values of $\varphi_1(\xi, t)$ and $\varphi_2(\xi, t)$, we obtain from (10), (11) the following expressions at discrete points $\xi_0, \xi_1, \ldots, \xi_S$, using the trapezium rule [9–11]:

$$\begin{gathered}
u = -\frac{\alpha(x)}{2\omega(x) m(x) S} \sum_{j=0}^{S} \sum_{i=1}^{2} d_j [v_{i2}(\xi_j, x, t) + v_{2i}(x, \xi_j, t)]\varphi_i(\xi_j, t), \\
u_S = -\frac{1}{2\omega_S m(1) S} \sum_{j=0}^{S} \sum_{i=1}^{2} d_j [v_{i2}(\xi_j, 1, t) + v_{2i}(1, \xi_j, t)]\varphi_i(\xi_j, t),
\end{gathered} \tag{12}$$

$$d_0 = d_S = 1/2, \qquad d_j = 1, \qquad j \neq 0, S.$$

The control forces are proportional to the deviations $\varphi_1(\xi_j, t)$ and $\varphi_2(\xi_j, t)$ $(j = 0, 1, 2, \ldots, S)$.

Regulator (10), (11) has a variable amplification coefficient in terms of the x axis and time. Equations (10) and (11) are not only the optimal equations of the regulator but also determine the regulator structure. These equations yield the law of formation of the control action based on the distributed values of deviations $\varphi_1(x,t)$ and their velocities $\varphi_2(x,t)$ at a given time.

If the law of control distribution along the body axis is given and we have to determine the optimal dependence on time, Eq. (10) should be replaced by an equation of the following form [9–11]:

$$u = -\frac{1}{2\omega_0}\int_0^1 \sum_{i=1}^2 \varphi_i(\xi,t)\int_0^1 \frac{\alpha(x)}{m}[v_{i2}(\xi,x,t)+v_{2i}(x,\xi,t)]\,dx\,d\xi, \qquad \omega_0 = \int_0^1 \omega(x)\,dx > 0. \tag{13}$$

According to (1), the law of axial distribution of the control forces has the form $\alpha(x)u(t)/m(x)$. If $\alpha = 0$ for $x \in [0,a)$ and $\alpha/m = 1$ for $x \in [a,1)$, then the control $u(t)$ is uniformly distributed over the section $x \in [a,1]$ of the axis of the flying vehicle. The control force is absent outside this sector of the axis.

The symmetric functions $v_{ij} = v_{ij}(x,\xi,t)$ that enter control (10), (12), (13), as it follows from the condition $K = 0$, should satisfy the following system of nonlinear integrodifferential equations [9–11, 14] under the final and boundary conditions (8), (9):

$$\begin{gathered}
\frac{\partial v_{11}}{\partial t} - \frac{\partial^2}{\partial x^2}\Big(EI\,\frac{\partial^2(v_{21}/m)}{\partial x^2}\Big) - \frac{\partial}{\partial x}\Big(q\,\frac{\partial(v_{21}/m)}{\partial x}\Big) + \frac{\partial(b_1v_{21}/m)}{\partial x} - \frac{\partial^2}{\partial \xi^2}\Big(EI\,\frac{\partial^2(v_{12}/m)}{\partial \xi^2}\Big) \\
- \frac{\partial}{\partial \xi}\Big(q\,\frac{\partial(b_1v_{12}/m)}{\partial \xi}\Big) + \frac{\partial(b_1v_{12}/m)}{\partial \xi} + w_{11} - R_{11} - \bar R_{11} = 0, \\
\frac{\partial v_{12}}{\partial t} + v_{11} - \frac{a_1(\xi)}{m(\xi)}\,v_{12} - \frac{\partial^2}{\partial x^2}\Big(EI\,\frac{\partial^2(v_{22}/m)}{\partial x^2}\Big) - \frac{\partial}{\partial x}\Big(q\,\frac{\partial(v_{22}/m)}{\partial x}\Big) + \frac{\partial(b_1v_{22}/m)}{\partial x} - R_{12} - \bar R_{12} = 0, \\
\frac{\partial v_{21}}{\partial t} + v_{11} - \frac{a_1(x)}{m(x)}\,v_{21} - \frac{\partial^2}{\partial \xi^2}\Big(EI\,\frac{\partial^2(v_{22}/m)}{\partial \xi^2}\Big) - \frac{\partial}{\partial \xi}\Big(q\,\frac{\partial(v_{22}/m)}{\partial \xi}\Big) + \frac{\partial(b_1v_{22}/m)}{\partial \xi} - R_{21} - \bar R_{21} = 0, \\
\frac{\partial v_{22}}{\partial t} + v_{12} + v_{21} - \Big[\frac{a_1(x)}{m(x)} + \frac{a_1(\xi)}{m(\xi)}\Big]v_{22} + w_{22} - R_{22} - \bar R_{22} = 0.
\end{gathered} \tag{14}$$

In the case of control (10), R_{ij} has the form

$$R_{ij} = \frac{1}{4}\int_0^1 \frac{\alpha^2(\eta) r_{ij}(t,\xi,\eta,x,y)}{\omega(\eta)m^2(\eta)}\,d\eta,$$

and in the case of control (13), it has the form

$$R_{ij} = \frac{1}{4\omega_0^2}\int_0^1\int_0^1 \frac{\alpha(y)\alpha(\eta)}{m(y)m(\eta)}\,r_{ij}(t,\xi,\eta,x,y)\,dy\,d\eta, \qquad \bar R_{ij} = -\frac{1}{4\omega_S m^2(1)}\,r_{ij}(t,\xi,1,x,1),$$

$$r_{11}(t,\xi,\eta,x,y) = [v_{12}(\xi,\eta,t) + v_{21}(\eta,\xi,t)][v_{12}(x,y,t) + v_{21}(y,x,t)],$$

$$r_{21}(t,\xi,\eta,x,y) = [v_{12}(\xi,\eta,t) + v_{21}(\eta,\xi,t)][v_{22}(x,y,t) + v_{22}(y,x,t)],$$

$$r_{12}(t,\xi,\eta,x,y) = [v_{22}(\xi,\eta,t) + v_{22}(\eta,\xi,t)][v_{12}(x,y,t) + v_{21}(y,x,t)],$$

$$r_{22}(t,\xi,\eta,x,y) = [v_{22}(\xi,\eta,t) + v_{22}(\eta,\xi,t)][v_{22}(x,y,t) + v_{22}(y,x,t)].$$

It is problematic to find the exact solution for system (14) because of its nonlinearity and variable coefficients. Therefore, to solve the problem of technical stability of the initial process (1)–(6) with (10), (11), we apply the method of comparison [1–8, 17, 18]. We assume that process (1)–(6), (10), (11) is put into correspondence to the Lyapunov functional $V[\varphi,t]$, which is discussed in more detail below.

We assume that the controlled process considered is technically stable in accordance with the following definitions.

Definition 1. The controlled dynamic process described by the boundary-value problem (1)–(6), (10), (11) is called technically stable on a given limited time interval $T_1 \subset I_1$ in terms of a given measure $\rho[\varphi]$ and Lyapunov functional $V[\varphi, t]$ if the following condition is satisfied for the functional $V[\varphi, t]$ positively defined with respect to the measure $\rho[\varphi]$ in the case of admissible continuous control u of the form (10), (11) along the disturbed solutions $\varphi(x,t)$ of the boundary-value problem (1)–(6), (10), (11):

$$V[\varphi(x,t),t] \leqslant A(t) \qquad \forall\, t \in T_1, \quad \forall\, x \in D,$$

if, at the initial time t_0, we have the inequality

$$V[\varphi_0(x),t_0] \leqslant b, \qquad t_0 \in T_1 \quad \forall\, x \in D, \tag{15}$$

where the value of $V[\varphi_0(x),t_0]$ is determined on the initial data (6), and the preliminarily chosen constant $b > 0$ and the limited function $A(t)$ specified on the time interval $T_1 \subset I_1$ satisfy the conditions $A(t) \leqslant \eta(t)$, $A(t_0) \geqslant b$, $0 < A(t) \leqslant \bar{A}$, $\bar{A} = \text{const} > 0$, $0 < \eta(t) \leqslant \tilde{\eta}$, $\tilde{\eta} = \text{const} > 0$, $t_0 \in T_1$, and $t_1 \in T_1$.

Definition 2. If the conditions of Definition 1 are satisfied at an arbitrary time interval $T_1 \subseteq I_1$, then the controlled dynamic process (1)–(6), (10), (11) is called technically stable in terms of the specified measure $\rho[\varphi]$ and Lyapunov functional $V[\varphi, t]$ on an infinite time interval I_1.

Definition 3. If the condition

$$\lim_{t\to+\infty} V[\varphi(x,t),t] = 0$$

is valid along the solutions of the boundary-value problem (1)–(6), (10), (11) in addition to the satisfied conditions of Definition 2, then, the initial controlled dynamic process (1)–(6), (10), (11) is called technical asymptotically stable in terms of the specified measure $\rho[\varphi]$ and Lyapunov functional $V[\varphi, t]$.

Definition 4. A controlled dynamic process with a variable structure (1)–(6), (10), (11) is called technically unstable in terms of the measure $\rho[\varphi]$ and specified Lyapunov functional $V[t, \varphi]$ in the region T_1 or I_1 for a given constant b and functions $A(t)$ and $\eta(t)$, if under satisfied condition (15) determined on the initial data (6), for the solutions $\varphi(x,t)$ of the boundary-value problem (1)–(6), (10), (11) for admissible control u (10), (11), there is a value $t_1 \in T_1$ or $t_1 \in I_1$ ($t_1 > t_0$) such that the following inequality is satisfied:

$$V[\varphi(x,t_1),t_1] > \tilde{\eta} \qquad (\tilde{\eta} = \text{const} > 0).$$

It follows from definitions 1–4 that the conditions of technical stability are characterized not only by the fact that a specified control process with distributed parameters is considered on a prior defined limited time interval, but also by the fact that the restrictions on the initial states of the initial process are independent of the conditions of majorization of the subsequent states of the controlled process on a given time interval. The unnecessary condition of negative determinacy or nonpositiveness of the total derivative of the Lyapunov functional on the states of the initial process, in contrast to stability according to Lyapunov, extends the range of the parameters of the initial process [1, 3–8].

Conditions of Technical Stability of Dynamic States of an Elastic Flying Vehicle on a Given Time Interval. To study the properties of technical stability of the process considered, we set the functional [7]

$$V[\varphi,t] = \int_0^1 \Big[\Big(\frac{\partial^2\varphi_1}{\partial x^2}\Big)^2 - P\Big(\frac{\partial\varphi_1}{\partial x}\Big)^2 + \Big(\frac{\partial\varphi_1}{\partial t}\Big)^2\Big]\,dx,$$
$$P = \bar{Q}_1 + a_1^0 + b_1^0, \qquad \bar{Q}_1 = \sup_x(\bar{Q}), \qquad a_1^0 = \sup_x(a_1), \qquad b_1^0 = \sup_x(b_1). \tag{16}$$

The following problem is posed: for given optimal control u (10), u_S (11) that satisfies relations (8), (9) and Eqs. (14), we have to determine conditions that ensure the fulfillment of the property

$$\varphi(x,t) \in \Omega(t), \qquad t \in T_1, \quad x \in D$$

with respect to the measure $\rho = \rho[\varphi]$ (7) for the solutions $\varphi(x,t)$ of problem (1)–(6), (10), (11) for given initial values

$$\varphi_0(x) \in \Omega_0, \qquad t_0 \in T_1 \quad \forall\, x \in D.$$

For the functional V (16), we obtain the following estimate from below:

$$3V[\varphi,t] \geqslant \pi^2 \int_0^1 dx\Big(\frac{\partial\varphi_1}{\partial x}\Big)^2 - P\int_0^1 dx\Big(\frac{\partial\varphi_1}{\partial x}\Big)^2 + 2\int_0^1 dx\Big(\frac{\partial^2\varphi_1}{\partial x^2}\Big)^2 - \frac{2}{\pi^2}P\int_0^1 dx\Big(\frac{\partial^2\varphi_1}{\partial x^2}\Big)^2$$

$$+ (1-P)\int_0^1 dx\Big(\frac{\partial\varphi_1}{\partial t}\Big)^2 \geqslant (1-P)\Big[\sup_x(\varphi_1)^2 + \sup\Big(\frac{\partial\varphi_1}{\partial x}\Big)\Big] + \int_0^1 dx\Big(\frac{\partial^2\varphi_1}{\partial x^2}\Big)^2$$

$$- \frac{1}{\pi^2}P\int_0^1 dx\Big(\frac{\partial^2\varphi_1}{\partial x^2}\Big)^2 + (1-P)\int_0^1 dx\Big(\frac{\partial\varphi_1}{\partial t}\Big)^2$$

$$\leqslant (1-P)\Bigg[\sup(\varphi_1)^2 + \sup_x\Big(\frac{\partial\varphi_1}{\partial x}\Big)^2 + \int_0^1 dx\Big(\frac{\partial^2\varphi_1}{\partial x^2}\Big)^2 + \int_0^1 dx\Big(\frac{\partial\varphi_1}{\partial t}\Big)^2\Bigg] \geqslant (1-P)\rho(\varphi).$$

Hence, we obtain

$$V[\varphi,t] \geqslant (1/3)(1-P)\rho(\varphi). \tag{17}$$

According to (17), the functional $V[\varphi,t]$ (16) is positively determined under the condition $0 < 1-P \leqslant 1$.

For the dynamic processes considered, it makes sense to consider the case

$$0 < 1-P < 1. \tag{18}$$

The quantity $\mu = 1-P$ is a small positive parameter: $\mu \in (0,1)$. Condition (18) is valid if the following inequality is satisfied [9–11]:

$$l(\tilde{Q} + \tilde{a}\sqrt{gl} + \tilde{b}) < G \qquad \forall\ x \in D. \tag{19}$$

Using the parameter μ determined in accordance with (17)–(19), we set a finite time interval T_1, where, in accordance with (1), we consider the dynamic behavior of the system: $T_1 = [t_0, N\mu^{-1}]$, where $t_0 \geqslant 0$ and $N = \text{const} > 0$ is a quantity that characterizes the reliability of the system.

Let functions $A(t)$ and $\eta(t)$ of the form

$$A(t) = \frac{M}{2}\exp\Big(-\frac{1}{\mu+t}\Big)\Big[\exp\Big(\frac{2}{\mu+t_0}\Big) - \exp\Big(\frac{2}{\mu+t}\Big)\Big] + y_0\exp\Big(-\frac{1}{\mu+t}\Big)\exp\Big(\frac{1}{\mu+t_0}\Big),$$

$$\eta(t) = \exp\Big(-\frac{1}{\mu+t}\Big)\Big\{\frac{\tilde{M}}{2}\Big[\exp\Big(\frac{2}{\mu+t_0}\Big) - \exp\Big(\frac{2}{\mu+t}\Big)\Big] + b\exp\Big(\frac{1}{\mu+t_0}\Big)\Big\} \leqslant \tilde{\eta}, \tag{20}$$

$$M, \tilde{M}, \tilde{\eta} = \text{const} > 0, \qquad M \leqslant \tilde{M}$$

be specified under the conditions

$$0 < y_0 \leqslant b, \qquad b = \text{const} > 0, \qquad y_0 = \text{const} > 0. \tag{21}$$

Using the functional V (16), we determine the set [1, 2, 8]

$$C_{y_0} = \{\varphi\colon\quad V[\varphi,t] \leqslant y_0 \quad \forall\ t \in T_1,\ \forall\ x \in D\},$$

which is assumed to satisfy the condition

$$\Omega_0 \subset C_{y_0} \quad \text{for} \quad t = t_0. \tag{22}$$

We calculate the total derivative of the functional V (16) with respect to t along the solutions of the boundary-value problem (1)–(6) with control (10), (11):

$$\frac{dV[\varphi(x,t),t]}{dt} = \frac{m(1)+EI(1)}{EI(1)m^2(1)}\,\varphi_2(t,1)\,\frac{1}{\omega_S}\int_0^1 \sum_{i=1}^{2}[v_{i2}(\xi,1,t)+v_{2i}(\xi,1,t)]\varphi_i(\xi,t)\,d\xi$$

$$+ 2\int_0^1 dx\Bigg[\Big(\frac{1}{m}\frac{\partial}{\partial x}\Big(EI(x)\frac{\partial^2\varphi_1(x,t)}{\partial x^2}\Big) - P\frac{\partial\varphi_1(x,t)}{\partial x}\Big)\frac{\partial\varphi_2(x,t)}{\partial x}$$

$$+\Big(\frac{\partial^4\varphi_1(x,t)}{\partial x^4}-\frac{a_1(x)}{m(x)}\,\varphi_2(x,t)-\frac{b_1(x)}{m(x)}\,\frac{\partial\varphi_1(x,t)}{\partial x}-\frac{1}{m(x)}\,\bar{Q}(x)\Big)\varphi_2(x,t)$$

$$-\frac{\alpha^2(x)}{2\omega(x)m^2(x)}\,\varphi_2(x,t)\int_0^1\sum_{i=1}^{2}[v_{i2}(\xi,x,t)+v_{2i}(x,\xi,t)]\varphi_i(\xi,t)\,d\xi\Bigg]. \tag{23}$$

We consider system (1)–(6), (10), (11) in the given domain

$$\bar{\Omega}=\Bigg\{t,x,\varphi_i,\frac{\partial\varphi_i}{\partial t},\frac{\partial^k\varphi_1}{\partial x^k},\frac{\partial^i\varphi_2}{\partial x^i},m(x),\bar{Q}(x),P,a_1,b_1,\omega,v_{ij},EI(x),\frac{\partial(EI)}{\partial x}:$$

$$t\in T_1,\ x\in D,|\varphi_i|\leqslant n_i,\ \Big|\frac{\partial\varphi_i}{\partial t}\Big|\leqslant l_i,\ \Big|\frac{\partial^k\varphi_1}{\partial x^k}\Big|\leqslant c_k,\ \Big|\frac{\partial^i\varphi_2}{\partial x^i}\Big|\leqslant\gamma_i,\ m_{\min}\leqslant m(x)\leqslant m_{\max},$$

$$0\leqslant\bar{Q}(x)\leqslant\bar{Q}_{\max},\ 0\leqslant P<1,\ a_{1\,\min}\leqslant a_1\leqslant a_{1\,\max},\ b_{1\,\min}\leqslant b_1\leqslant b_{1\,\max},$$

$$0<\omega_{\min}\leqslant\omega\leqslant\omega_{\max},\ |v_{ij}|\leqslant\theta_{ij},\ 0<EI\leqslant K_1\equiv\max_x(EI),\ \Big|\frac{\partial(EI)}{\partial x}\Big|\leqslant K_2;$$

$$c_k,\ \gamma_i,\ m_{\min},\ m_{\max},\ b_{1\,\min},\ b_{1\,\max},\ \bar{Q}_{\max},\ \omega_{\min},\ \omega_{\max},\ K_1,\ K_2=\text{const}>0,i=1,\,2,\ j=1,\,2,\ k=1,\ldots,4\Bigg\}.$$

We denote the expression in the right side of equality (23) as $M(t)$. We consider the function

$$\bar{\Phi}(t)=M(t)-(\mu/(3(\mu+t)^2))\rho(\varphi(x,t)). \tag{24}$$

Let the function $\bar{\Phi}(t)$ (24) satisfy the condition

$$|\bar{\Phi}(t)|\leqslant\Phi(t)\equiv M\exp(1/(\mu+t))/(\mu+t)^2, \tag{25}$$

where $M=\text{const}>0$ is a prescribed quantity. In particular, $|M(t)|\leqslant M$,

$$M\equiv\frac{m(1)+EI(1)}{EI(1)m^2(1)}\,n_2\,\frac{1}{\omega_S}\sum_{i=1}^{2}[\theta_{i2}+\theta_{2i}]n_i+2\Big[\Big(\frac{1}{m_{\min}}\,K_2c_2+\frac{1}{m_{\min}}\,K_1c_3+c_1\Big)\gamma_1$$

$$+\Big(c_4+\frac{a_{1\,\max}}{m_{\min}}\,n_2+\frac{b_{1\,\max}}{m_{\min}}\,c_1+\frac{\bar{Q}_{\max}}{m_{\min}}\Big)n_2+\frac{1}{2\omega_{\min}m_{\min}^2}\,c_2\sum_{i=1}^{2}[\theta_{i2}+\theta_{2i}]n_i\Big]\leqslant\tilde{M}.$$

In the region T_1, there is an integral $\sigma(t)=\int_{t_0}^{t}\Phi(\tau)\,d\tau$. We consider the function $z(t)=V[\varphi(x,t),t]-\sigma(t)$ along the solutions of problem (1)–(6), (10), (11). Under the above conditions, the estimates for dV/dt along the solutions of this problem yield the following inequality [7, 15, 17, 18]:

$$\frac{dz(t)}{dt}\leqslant\frac{\mu}{(\mu+t)^2}\,[z(t)+\sigma(t)]. \tag{26}$$

Equation (26) leads to the Cauchy problem of comparison of the form [2–8, 17]

$$\frac{dy}{dt}=\frac{1}{(\mu+t)^2}\,[y+\sigma(t)],\qquad t\in T_1; \tag{27}$$

$$y(t_0)=y_0\geqslant V_0\equiv V[\varphi_0(x),t_0],\qquad t_0\in T_1\quad\forall\ x\in D. \tag{28}$$

Under the above conditions, problem (27), (28) in the region T_1 has a continuous solution

$$y(t)=\frac{M}{2}\,\exp\Big(-\frac{1}{\mu+t}\Big)\Big[\exp\Big(\frac{2}{\mu+t_0}\Big)-\exp\Big(\frac{2}{\mu+t}\Big)\Big]+y_0\exp\Big(\frac{1}{\mu+t_0}\Big)\exp\Big(-\frac{1}{\mu+t}\Big)-\sigma(t). \tag{29}$$

Using (29) and the corresponding theorem of differential inequalities [17, 18], we find

$$z(t)<y(t),\qquad t\in T_1. \tag{30}$$

From (30), along the solution of problem (1)–(6), (10), (11) under conditions (18), (19), we obtain

$$V(t) \leqslant y(t) + \sigma(t), \qquad t \in T_1. \tag{31}$$

From (29) and (31), we obtain the inequalities

$$V(t) \leqslant A(t) \leqslant \eta(t), \qquad A(t) \equiv \bar{y}(t) + \sigma(t). \tag{32}$$

For $t \in T_1$, we have

$$A(t) \leqslant \frac{M}{2} \exp\Big(-\frac{1}{\mu + N\mu^{-1}}\Big)\Big[\exp\Big(\frac{2}{\mu + t_0}\Big) - \exp\Big(\frac{2}{\mu + N\mu^{-1}}\Big)\Big] + y_0 \exp\Big(-\frac{1}{\mu + N\mu^{-1}}\Big) \exp\Big(\frac{1}{\mu + t_0}\Big),$$

$$\eta(t) \leqslant \exp\Big(-\frac{1}{\mu + N\mu^{-1}}\Big)\Big\{\frac{M}{2}\Big[\exp\Big(\frac{2}{\mu + t_0}\Big) - \exp\Big(\frac{2}{\mu + N\mu^{-1}}\Big)\Big] + b \exp\Big(\frac{1}{\mu + t_0}\Big)\Big\} \leqslant \tilde{\eta}, \tag{33}$$

$$A(t_0) \equiv b, \qquad V_0 \leqslant b$$

along the solution of problem (1)–(6), (10), (11) under conditions (18) and (19).

From inequalities (32) and (33), we obtain [2, 18]

$$C_{A(t)} \subset \Omega(t), \qquad C_{A(t)} = \{\varphi\colon \ \ V[\varphi, t] \leqslant A(t) \quad \forall\, t \in T_1,\ \forall\, x \in D\}. \tag{34}$$

It follows from relation (34) and condition (22) with allowance for (18), (19), (21), and (28) that the initial process (1)–(6) for $\varphi_0 \in \Omega_0$, control (10), (11), and conditions (8), (9), and (14) is technically stable in terms of the specified measure ρ (7) and Lyapunov functional V (16) on the limited time interval T_1 [1–8].

For $t \to +\infty$ and condition (20), the estimate $A(t) \leqslant \eta(t)$ is valid. The estimate

$$A(t) \leqslant A, \qquad A \equiv (M/2)[\exp(2/(\mu + t_0)) - 1] + y_0 \exp(1/(\mu + t_0))$$

is valid for all $T_1 \subseteq I_1$, which follows from (32) for $t \to +\infty$. Thus, process (1)–(6), (10), (11) for $\varphi_0 \in \Omega_0$ is technically stable on an infinite time interval I_1 in terms of the specified measure ρ and Lyapunov functional V. For a given function $\Phi(t)$ of the form (25), the condition of asymptotic technical stability of the initial process is not satisfied [1, 3–8, 18].

These conditions of technical stability of the initial controlled process (1)–(6), (10), (11) are violated if the parameters Q_0, $\tilde{a}$, and $\tilde{b}$ satisfy the inequality $P \geqslant 1$, which, in accordance with (16), (18), (19), is identical to the inequality [1, 3–8]

$$(l/G)(Q_0 + \tilde{a}\sqrt{gl} + \tilde{b}) \geqslant 1 \qquad \forall\, x \in D, \tag{35}$$

since the condition of positive determinacy (17) for functional (16) is not satisfied in this case. The initial system (1)–(6) for $\varphi_0 \in \Omega_0$, control (10), (11), and conditions (8), (9), and (14) is technically unstable in T_1 or I_1 in terms of the measure ρ and Lyapunov functional V, if the function $A(t)$ in these regions satisfy the condition

$$A(t) \to +\infty \qquad \text{for} \quad t \in T_1 \quad \text{or} \quad t \in I_1. \tag{36}$$

In particular, condition (36) is satisfied for $t_0 = 0$ and arbitrary values of $t \geqslant 0$, as $\mu \to 0$, which, as follows from conditions (18) and (19), corresponds to a drastic increase in parameters that characterize the initial controlled process (1)–(6), (10), (11), for instance, a drastic increase in the increment of the transverse load Q_0 due to the curvature of the longitudinal axis of the system in vertical flight. In the latter case, the critical increment Q_0^{cr} of the transverse load due to the curvature of the longitudinal axis of the system is determined using inequality (35):

$$Q_0^{cr} = G/l - \tilde{a}\sqrt{gl} - \tilde{b}. \tag{37}$$

Equation (37) indicates an explicit dependence of the increment of the transverse load on a vertically moving system on the other governing parameters.

Let the following condition be valid under the above properties (16)–(29):

$$\theta^{-1} A(t) \leqslant \eta(t), \qquad \theta = (1 - P)/3. \tag{38}$$

In particular, property (38) is valid under the assumption that the following inequalities are satisfied:

$$\theta^{-1} z_0 \leqslant b, \qquad \theta^{-1} M \leqslant \tilde{M}. \tag{39}$$

Then along the solution of problem (1)–(6), (10), (11) with respect to the measure ρ (7), we obtain the estimate

$$\rho[\varphi(x,t)] \leqslant \eta(t), \qquad t \in T_1 \subseteq I_1. \tag{40}$$

Hence, under the above assumptions (7), (15)–(34) and additional conditions (38), (39), the initial process (1)–(6), (10), (11) for $\forall\, t \in T_1 \subseteq I_1$ and property (40) being satisfied is technically stable in terms of the given measure ρ (7) on finite or infinite time intervals [1, 3–8, 17].

Conditions of Asymptotic Technical Stability of an Elastic Flying System. For control u (10), u_S (11) and conditions (8), (9), and (14), we consider the functions

$$\Phi_1(t,\varphi(x,t),u_S) = \frac{m(1)+EI(1)}{EI(1)m^2(1)}\,\varphi_2(1,t)\,\frac{1}{\omega_S}\int_0^1 \sum_{i=1}^2 [v_{i2}(\xi,1,t)+v_{2i}(1,\xi,t)]\varphi_i(\xi,t)\,d\xi$$

$$+\,2\int_0^1 dx\Big[\Big(\frac{1}{m}\,\frac{\partial}{\partial x}\Big(EI(x)\,\frac{\partial^2\varphi_1(x,t)}{\partial x^2}\Big)\Big)\frac{\partial\varphi_2(x,t)}{\partial x} + \frac{\partial^4\varphi_1(x,t)}{\partial x^4}\,\varphi_2(x,t)\Big]; \tag{41}$$

$$\Phi_2(t,\varphi(x,t),u) = -2\int_0^1 dx\Bigg[P\,\frac{\partial\varphi_1(t,x)}{\partial x}\,\frac{\partial\varphi_2(t,x)}{\partial x} + \frac{a_1(x)}{m(x)}\,\varphi_2^2(t,x)$$

$$+\,\frac{b_1(x)}{m(x)}\,\frac{\partial\varphi_1(t,x)}{\partial x}\,\varphi_2(t,x) + \frac{1}{m(x)}\,\bar{Q}(x)\varphi_2(x,t) + \frac{\alpha^2(x)}{2\omega(x)m^2(x)}\,\varphi_2(x,t)\int_0^1 \sum_{i=1}^2 [v_{i2}(\xi,x,t)+v_{2i}(x,\xi,t)]\varphi_i(\xi,t)\,d\xi\Bigg],$$

assuming that they exist in the region I_1. Let the following conditions be satisfied for the functions $\Phi_1(t,\varphi(x,t),u_S)$ and $\Phi_2(t,\varphi(x,t),u)$ if the properties (8), (9), and (14) are valid on the solution of the initial problem (1)–(6), (10), (11):

$$\Phi_2(t,\varphi(x,t),u) \leqslant -V[\varphi(x,t),t], \qquad t \in T_1 \subseteq I_1; \tag{42}$$

$$|\Phi_1(t,\varphi(x,t),u_S)| \leqslant M m_1(t), \qquad t \in T_1 \subseteq I_1. \tag{43}$$

Here $m_1(t)$ is a limited function for which the following integrals exist:

$$\sigma(t) = M\int_{t_0}^{t} m_1(\tau)\,d\tau, \quad t \in T_1 \subseteq I_1, \qquad \sigma_1(t) = \int_{t_0}^{t} \mathrm{e}^{\tau} m_1(\tau)\,d\tau, \quad t \in T_1 \subseteq I_1. \tag{44}$$

Under conditions (44), we use the following functions $A(t)$ and $\eta(t)$:

$$A(t) = \mathrm{e}^{-(t-t_0)}\Big[y_0 + M\mathrm{e}^{-t_0}\int_{t_0}^{t} \mathrm{e}^{\tau} m_1(\tau)\,d\tau\Big],$$

$$\eta(t) = \mathrm{e}^{-(t-t_0)}\Big[\tilde{M}\mathrm{e}^{-t_0}\int_{t_0}^{t} \mathrm{e}^{\tau} m_1(\tau)\,d\tau + b\Big], \quad M \leqslant \tilde{M}. \tag{45}$$

Using the inequality of the form

$$\frac{dV(t)}{dt} \leqslant -V(t) + M m_1(t) \qquad (t \in T_1 \subseteq I_1),$$

along the solutions of problem (1)–(6), (10), (11), we consider the system of comparison

$$\frac{dy}{dt} = -[y + \sigma(t)], \qquad t \in T_1 \subseteq I_1; \tag{46}$$

$$y(t_0) = y_0 \geqslant V_0 \equiv V[\varphi_0(x),t_0], \qquad 0 < y_0 \leqslant b, \quad t \in T_1 \subseteq I_1 \quad \forall\, x \in D. \tag{47}$$

The solution for (46) and (47) has the form

$$\bar{y}(t) = \mathrm{e}^{-(t-t_0)}\Big[y_0 + M\mathrm{e}^{-t_0}\int_{t_0}^{t} \mathrm{e}^{\tau} m_1(\tau)\,d\tau\Big] - \sigma(t).$$

If Eqs. (44) and (45) are valid along the solutions of system (1)–(6), (10), (11), we obtain

$$V(t) \leqslant A(t) \leqslant \eta(t), \qquad A(t) \equiv \bar{y}(t) + \sigma(t). \tag{48}$$

If the function $A(t)$ (45) is limited as $t \to +\infty$, then process (1)–(6), (10), (11) is technically stable on the unlimited time interval I_1 in terms of the measure ρ and Lyapunov functional V. If we have $\lim\limits_{t\to+\infty} A(t) = 0$ for the functions $\sigma(t)$ and $\sigma_1(t)$ (44), then process (1)–(6), (10), (11) is asymptotically technically stable in terms of the given measure ρ and Lyapunov functional V.

We consider various cases of the behavior of the function $m_1(t)$ as $t \to \infty$. Let in (42) and (43) we have

$$m_1(t) = \mathrm{e}^{-nt} \to 0 \quad \text{as} \quad t \to +\infty, \quad n \geqslant 2 \tag{49}$$

(n is a natural number). Then, we use the functions $A(t)$ and $\eta(t)$ of the following form in Eq. (48) for problem (1)–(6), (10), (11):

$$A(t) = \mathrm{e}^{-(t-t_0)}\Big[y_0 + M\mathrm{e}^{-t_0}\,\frac{1}{n-1}\Big(\mathrm{e}^{-(n-1)t_0} - \mathrm{e}^{-(n-1)t}\Big)\Big],$$
$$\eta(t) = \mathrm{e}^{-(t-t_0)}\Big(b + \tilde{M}\,\frac{1}{n-1}\,\mathrm{e}^{-nt_0}\Big) \leqslant \tilde{\eta}, \quad M \leqslant \tilde{M}. \tag{50}$$

If $m_1(t) = \exp(1/(\mu+t)-t)/(\mu+t)^2 \to 0$ as $t \to +\infty$ and for $\mu \in (0,1)$, then we use the following functions as $A(t)$ and $\eta(t)$ in (48):

$$A(t) = \mathrm{e}^{-(t-t_0)}\Big\{y_0 + M\mathrm{e}^{-t_0}\Big[\exp\Big(\frac{1}{\mu+t_0}\Big) - \exp\Big(\frac{1}{\mu+t}\Big)\Big]\Big\},$$
$$\eta(t) = \mathrm{e}^{-(t-t_0)}\Big[b + \tilde{M}\exp\Big(\frac{1}{\mu+t_0} - t_0\Big)\Big] \leqslant \tilde{\eta}, \quad M \leqslant \tilde{M}. \tag{51}$$

According to Definition 3, relations (48)–(51) ensure asymptotic technical stability of the unsteady nonlinear elastic controlled process (1)–(6), (10), (11) in terms of the given measure ρ and Lyapunov functional V. We can easily see that, if relations (42)–(44) are satisfied, estimate (40) with respect to the measure ρ (7) is also valid under conditions (38), (39) on the solution of the initial process.

If the functions $\Phi_1(t,\varphi(x,t),u_S)$ and $\Phi_2(t,\varphi(x,t),u)$ satisfy the condition

$$\Phi_1(t,\varphi(x,t),u_S) \leqslant -\Phi_2(t,\varphi(x,t),u) \qquad \forall\, t \in I_1, \tag{52}$$

then according to (23), the controlled dynamic process (1)–(6), (10), (11) is stable according to Lyapunov. Hence, the above-formulated sufficient conditions of technical stability of the initial controlled process (1)–(6), (10), (11) on an infinite time interval include, in accordance with (41), (42), (52), conditions of stability of this process according to Lyapunov with respect to the measure ρ (7).

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